

# The Semiclassical Theory of Quantized Fields in Classical Electromagnetic Backgrounds

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## Abstract

We show the mathematical formulation of the process of pair production in electromagnetic fields for spinless particles. We compute the probability that  $n$  pairs are created in the semiclassical approximation, and we prove that, in this approach, the pair creation phenomenon is a stochastic Poisson process. Finally, we give a rigorous demonstration and a suitable interpretation of the Schwinger formula.

**Key words.** Pair production, Schwinger's formula, Semiclassical approach.

**AMS subject classifications.** 34E20, 81Q05, 81Q20.

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# 1 Introduction

The phenomenon of pair production in the presence of a classical electromagnetic background has been studied by some authors. Often, many of them say that particle-antiparticle creation is due to a kind of relativistic tunneling effect. This is stated, generally, when the electromagnetic potential is time-independent and has the following form:  $(V(z), 0, 0, 0)$ , i.e., the potential vector is zero, and therefore the field is purely electric.

These authors study, for spinless (spin) particles, the normal modes of the Klein-Gordon (Dirac) equation, which is, in this case, an ordinary differential equation in the variable  $z$ . The equation depends on the two first components of the momentum, namely  $p_{\perp} := (p_1, p_2)$ . When the electric field is spatially confined, these modes are asymptotically plane waves, and we can compute the transmission and reflection coefficient corresponding to the  $j$ -mode. Therefore, when the energy of the  $j$ -mode, namely  $E_j$ , verifies the inequality  $E_j - eV(z) > \sqrt{c^2 p_{\perp}^2 + m^2 c^4}$  in a region of  $\mathbb{R}$ , and  $eV(z) - E_j > \sqrt{c^2 p_{\perp}^2 + m^2 c^4}$  in another region, (i.e., when the kinetic energy in a region is greater than  $\sqrt{c^2 p_{\perp}^2 + m^2 c^4}$  and the kinetic energy is less than  $-\sqrt{c^2 p_{\perp}^2 + m^2 c^4}$  in another region), we have Klein's paradox. It is also well-known that, when Klein's paradox exists, the modulus of the transmission coefficient corresponding to the  $j$ -mode is interpreted as the relative probability that a pair is created in the  $j$ -state (see [8], [10], [17], [23]). This is the interpretation of the particle-antiparticle production using the relativistic tunneling effect.

However, there are authors who interpret the relativistic tunneling effect in another way. For example, in [26], the authors study particle production in purely time-independent magnetic fields. It is clear that, in this case, the energy of the system is only kinetic. Consequently, Klein's paradox is not present and, no pairs are created. However, in [26], the authors always interpret the transmission coefficient as the relative probability that a pair is produced. They then conclude that, in the presence of a purely time-independent magnetic field, pairs are created from the vacuum state. On the other hand, using Schwinger's interpretation ([10], [18], [19], [25], [26]), the exponential of twice the imaginary part of the effective action gives the probability that the vacuum state remains unchanged. However, since purely magnetic fields do not have imaginary part (see [26]), they reach the conclusion that no pairs are created in this situation. Consequently, the authors claim that a contradiction arises from the tunneling interpretation and Schwinger's interpretation. It is clear that, if we do not correctly interpret the relativistic tunneling effect, some type of contradiction appears.

In this paper, in order to avoid wrong interpretations, we prefer to approach pair production phenomenon using Dirac's point of view. For this reason, it is appropriate to first study the case of particles with  $\text{spin} \frac{1}{2}$ . Due to the Dirac Sea hypothesis and the Pauli Exclusion Principle, the vacuum state is this one in which all (Dirac would say nearly all [4]) the negative **kinetic** energy states are fulfilled. We focus on the word kinetic because it is the key to everything. In the same way, is this the state of a particle when all the negative kinetic energy states are fulfilled and, when one positive kinetic energy state is fulfilled. This is the state of an antiparticle when the negative kinetic energy states are fulfilled except one, etc.. ([5], [13]).

From this interpretation we can deduce very interesting consequences. For example, when the electromagnetic background is time-independent two different situations arise:

A) The scalar potential is zero, i.e., the field is purely magnetic. In this situation, the energy is kinetic only and, is constant over time. Consequently, the eigenfunctions of the energy operator are stationary states. There is therefore no particle creation.

B) The scalar potential is not zero. In this case, there is an electric field. The energy of the system is decomposed into kinetic and potential energy. Now, the states that describe a determinate number of particles and antiparticles, i.e., the eigenfunctions of the kinetic energy operator, are **not** eigenfunctions of the total energy of the system. Consequently, there is particle production. Physically, when a state with a determinate number of particles and antiparticles evolves, a part of its potential energy becomes transformed into kinetic

energy. This can become mass, and pair production appears.

Another important case appears when the potential does not depend of the spatial variables, i.e., there is only an electric field. Using a gauge transformation, it is easy to see that the system is equivalent to another system that only has kinetic energy. In this case, the energy operator depends on time and so does the eigenfunctions. For this reason, the state with a determinate number of particles and antiparticles not is stationary. Consequently, there is particle creation. Physically, when the kinetic energy of the system changes, part of it becomes mass, and pairs are produced.

For spinless particles we obtain similar results. Effectively, the kinetic energy is composed by an infinite number of harmonic oscillators with frequencies that depend on time when the vector potential is time-dependent. For this reason, we will find similar situations to the case of particles with spin  $\frac{1}{2}$ .

Through this interpretation, the way to compute any type of probability is clear. In fact, it can be demonstrated that, when the potential is spatially and temporally confined, the probability that the vacuum state remains unchanged is equal to the exponential of twice the imaginary part of the effective action. Proof of this result for particles with spin  $\frac{1}{2}$  is in [19]. In order to prove this result, the authors use the Perturbation Theory in all the orders. Following this demonstration step by step in the case of the Klein-Gordon field, we obtain that the result is also valid for spinless particles. We thus conclude that our interpretation coincides with Schwinger's interpretation of the exponential of twice the imaginary part of the effective action. Schwinger also interpreted (see [25]) that minus twice the imaginary part of the effective action is the probability that a pair is created. In this work, we can see that this interpretation is incorrect. Effectively, we see that minus twice the imaginary part of the effective action is, within the semiclassical limit, the average number of produced pairs.

The paper is organized as follows: In Section II, we study the production of spinless particles by uniform electric fields. Firstly, we use the Schrödinger picture. Secondly, we use the Heisenberg picture. Using the Bogolubov coefficients, we show the way to compute probabilities in the two pictures. At the end of the section we prove that, in the semiclassical approximation, pair production is a stochastic Poisson process, and we compute the average number of pairs produced.

In Section III, we study the spinless production by potentials that are spatially and temporally confined. We provide the method to obtain the kinetic energy decomposition in harmonic oscillators. Once we have obtained this decomposition, we introduce the creation and annihilation operators, that are time-dependent when the potential vector is time dependent. From these operators, we can construct the function corresponding to the states with a determinate number of particles and antiparticles.

Finally, in Section IV, we give rigorous proof and a suitable interpretation of the Schwinger formula.

## 2 Spinless particles in homogeneous fields

### 2.1 The Schrödinger Picture

In this section we consider the Klein-Gordon equation

$$-\hbar^2 \partial_t^2 \psi(\vec{x}, t) = (i\hbar c \vec{\nabla} - e\vec{f}(t))^2 \psi(\vec{x}, t) + m^2 c^4 \psi(\vec{x}, t),$$

in a box of volume  $L^3$ , with a periodic boundary condition.

In the Schrödinger picture, the Klein-Gordon equation is equivalent to a Hamiltonian system, composed of an infinite number of harmonic oscillators with frequencies which depend on time ([10], [12]).

The energy and the electric charge of the system are as follows:

$$E_S(t) = \frac{1}{2} \sum_{\vec{k} \in \mathbb{Z}^3} \left( P_{\vec{k}}^2(t) + \omega_{\vec{k}}^2(t) Q_{\vec{k}}^2(t) \right) + \left( \bar{P}_{\vec{k}}^2(t) + \omega_{\vec{k}}^2(t) \bar{Q}_{\vec{k}}^2(t) \right)$$

$$\rho_S(t) = \frac{1}{\hbar} \sum_{\vec{k} \in \mathbb{Z}^3} (\bar{Q}_{\vec{k}}(t) P_{\vec{k}}(t) - Q_{\vec{k}}(t) \bar{P}_{\vec{k}}(t)),$$

where  $\omega_{\vec{k}}(t) := \frac{1}{\hbar} \epsilon_{\vec{k}}(t) = \frac{1}{\hbar} \sqrt{c^2 \left| \frac{2\pi\hbar\vec{k}}{L} + \frac{e}{c} \vec{f}(t) \right|^2 + m^2 c^4}$  is the frequency.

In order to obtain the quantum theory we must quantize these oscillators, i.e. we make the replacement

$$Q_{\vec{k}}(t) \rightarrow \hat{Q}_{\vec{k}} := Q_{\vec{k}}; \quad \bar{Q}_{\vec{k}}(t) \rightarrow \hat{\bar{Q}}_{\vec{k}} := \bar{Q}_{\vec{k}}$$

$$P_{\vec{k}}(t) \rightarrow \hat{P}_{\vec{k}} := -i\hbar \partial_{Q_{\vec{k}}}; \quad \bar{P}_{\vec{k}}(t) \rightarrow \hat{\bar{P}}_{\vec{k}} := -i\hbar \partial_{\bar{Q}_{\vec{k}}},$$

and the quantum equation, in the Schrödinger picture, becomes

$$i\hbar \partial_t |\Phi\rangle_S = \frac{1}{2} \sum_{\vec{k} \in \mathbb{Z}^3} \left[ \left( -\hbar^2 \partial_{Q_{\vec{k}}}^2 + \omega_{\vec{k}}^2(t) Q_{\vec{k}}^2 \right) + \left( -\hbar^2 \partial_{\bar{Q}_{\vec{k}}}^2 + \omega_{\vec{k}}^2(t) \bar{Q}_{\vec{k}}^2 \right) \right] |\Phi\rangle_S - \sum_{\vec{k} \in \mathbb{Z}^3} \epsilon_{\vec{k}}(t) |\Phi\rangle_S.$$

The eigenfunctions of the energy and of the electric charge operators must now be found. Firstly, we must introduce the creation and annihilation operators for particles and antiparticles in the Schrödinger picture, at time  $t$  [12].

$$\begin{aligned} \hat{a}_{S,\vec{k}}(t) &= \frac{1}{2\sqrt{\epsilon_{\vec{k}}(t)}} \left[ \left( i\hat{P}_{\vec{k}} + \omega_{\vec{k}}(t) \hat{Q}_{\vec{k}} \right) + i \left( i\hat{\bar{P}}_{\vec{k}} + \omega_{\vec{k}}(t) \hat{\bar{Q}}_{\vec{k}} \right) \right] \\ \hat{b}_{S,-\vec{k}}^\dagger(t) &= \frac{1}{2\sqrt{\epsilon_{\vec{k}}(t)}} \left[ \left( -i\hat{P}_{\vec{k}} + \omega_{\vec{k}}(t) \hat{Q}_{\vec{k}} \right) + i \left( -i\hat{\bar{P}}_{\vec{k}} + \omega_{\vec{k}}(t) \hat{\bar{Q}}_{\vec{k}} \right) \right]. \end{aligned}$$

Then, using these operators, we obtain

$$\hat{E}_S(t) = \sum_{\vec{k} \in \mathbb{Z}^3} \epsilon_{\vec{k}}(t) \left( \hat{a}_{\vec{k}}^\dagger(t) \hat{a}_{\vec{k}}(t) + \hat{b}_{-\vec{k}}^\dagger(t) \hat{b}_{-\vec{k}}(t) \right); \quad \hat{\rho}_S(t) = \sum_{\vec{k} \in \mathbb{Z}^3} \left( \hat{a}_{\vec{k}}^\dagger(t) \hat{a}_{\vec{k}}(t) - \hat{b}_{-\vec{k}}^\dagger(t) \hat{b}_{-\vec{k}}(t) \right).$$

Now, we construct the vacuum state at time  $t$ . If we consider  $|0_{\vec{k},t}\rangle_S = \sqrt{\frac{\omega_{\vec{k}}(t)}{\pi\hbar}} \exp\left(-\frac{\omega_{\vec{k}}(t)}{2\hbar}(Q_{\vec{k}}^2 + \bar{Q}_{\vec{k}}^2)\right)$ , then the vacuum state at time  $t$ ,  $|0_t\rangle$ , is  $|0_t\rangle_S = \prod_{\vec{k} \in \mathbb{Z}^3} |0_{\vec{k},t}\rangle_S$ , since  $\hat{E}_S(t)|0_t\rangle_S = 0$  and  $\hat{\rho}_S(t)|0_t\rangle_S = 0$ .

Therefore, starting at the vacuum state and, using the creation operators, we can construct the Fock space.

## 2.2 The Heisenberg Picture, “in” and “out” formalism, and Bogolubov coefficients

In order to obtain the Heisenberg picture, we must first define  $\hat{E}_H(t) = T(0, t) \hat{E}_S(t) T(t, 0)$ , where  $T(t, 0)$  is the quantum evolution operator, i.e., it verifies

$$\begin{cases} i\hbar \dot{T}(t, 0) &= \hat{E}_S(t) T(t, 0) \\ T(0, 0) &= Id. \end{cases} \quad (1)$$

Let  $|\psi_t\rangle_S$  be an eigenfunction of the operator  $\hat{E}_S(t)$  with eigenvalue  $\lambda(t)$ , then  $|\psi_t\rangle_H := T(0, t)|\psi_t\rangle_S$  is an eigenfunction of the operator  $\hat{E}_H(t)$  with eigenvalue  $\lambda(t)$ . That is,  $T(0, t)$  apply the eigenfuctions of the energy operator in the Schrödinger picture to the eigenfuctions of the energy operator in the Heisenberg picture.

In this picture, the creation and annihilation operators are:

$$\hat{a}_{H,\vec{k}}(t) = T(0, t)\hat{a}_{S,\vec{k}}(t)T(t, 0), \dots, \hat{b}_{H,-\vec{k}}^\dagger(t) = T(0, t)\hat{b}_{S,-\vec{k}}^\dagger(t)T(t, 0).$$

Then, the “in” and “out” creation and annihilation operators are

$$\hat{a}_{(in),\vec{k}} := \lim_{t \rightarrow \mp\infty} \hat{a}_{H,\vec{k}}(t), \dots, \hat{b}_{(in),\vec{k}}^\dagger := \lim_{t \rightarrow \mp\infty} \hat{b}_{H,\vec{k}}^\dagger(t),$$

and the “in” and “out” vacuum state is

$$|0_{\vec{k},(in)}\rangle := \lim_{t \rightarrow \mp\infty} |0_{\vec{k},t}\rangle_H, \quad |0_{(in)}\rangle := \prod_{\vec{k}} |0_{\vec{k},(in)}\rangle = \lim_{t \rightarrow \mp\infty} |0_t\rangle_H.$$

In order to obtain the relationship between the creation and annihilation operators in the Heisenberg picture at different times, we define the Bogolubov coefficients in the following form [11]:

$$\begin{pmatrix} \hat{a}_{H,\vec{k}}(t_2) \\ \hat{b}_{H,\vec{k}}^\dagger(t_2) \end{pmatrix} = \begin{pmatrix} \alpha_{\vec{k}}^*(t_2, t_1) & \beta_{\vec{k}}(t_2, t_1) \\ \beta_{\vec{k}}^*(t_2, t_1) & \alpha_{\vec{k}}(t_2, t_1) \end{pmatrix} \begin{pmatrix} \hat{a}_{H,\vec{k}}(t_1) \\ \hat{b}_{H,\vec{k}}^\dagger(t_1) \end{pmatrix}$$

**Remark 2.1.** The Bogolubov coefficients verifies  $|\alpha_{\vec{k}}(t_2, t_1)|^2 - |\beta_{\vec{k}}(t_2, t_1)|^2 = 1$ .

It is easy to verify that (see [11], [14])

$$|0_{\vec{k},t_2}\rangle_H = c_{\vec{k}} \sum_{n=0}^{\infty} \left( -\frac{\beta_{\vec{k}}(t_2, t_1)}{\alpha_{\vec{k}}^*(t_2, t_1)} \right)^n |n_{\vec{k},t_1}\rangle_H; \quad |0_{\vec{k},t_1}\rangle_H = \bar{c}_{\vec{k}} \sum_{n=0}^{\infty} \left( \frac{\beta_{\vec{k}}(t_2, t_1)}{\alpha_{\vec{k}}(t_2, t_1)} \right)^n |n_{\vec{k},t_1}\rangle_H,$$

where  $|c_{\vec{k}}|^2 = |\bar{c}_{\vec{k}}|^2 = |\alpha_{\vec{k}}(t_2, t_1)|^{-2}$ , and  $|n_{\vec{k},t}\rangle_H$  is the vector, that a time  $t$ , contains  $n$  particles in the  $\vec{k}$ -state and  $n$  antiparticles in the  $-\vec{k}$ -state, i.e.  $|n_{\vec{k},t}\rangle_H = \frac{(\hat{a}_{H,\vec{k}}^\dagger(t))^n (\hat{b}_{H,-\vec{k}}^\dagger(t))^n}{n!} |0_{\vec{k},t}\rangle_H$ .

### 2.3 Probability Formulae

Let  $P_{n,\vec{k}}(t_2, t_1)$  be the probability that  $n$  pairs are created in the  $\vec{k}$ -state, after the evolution of the vacuum state from  $t_1$  to  $t_2$ . Then, in the Schrödinger and Heisenberg picture, the formulae that give this probability are:

$$P_{n,\vec{k}}(t_2, t_1) = |{}_S \langle n_{\vec{k},t_2} | T(t_2, t_1) | 0_{t_1} \rangle_S|^2 = |{}_H \langle n_{\vec{k},t_2} | 0_{t_1} \rangle_H|^2, \quad (2)$$

and for the average number of produced pairs in the  $\vec{k}$ -state at time  $t_2$  created from the vacuum state at time  $t_1$ , namely  $N_{\vec{k}}(t_2, t_1)$ , the formulae are

$$N_{\vec{k}}(t_2, t_1) = {}_S \langle 0_{t_1} | T(t_1, t_2) \hat{a}_{S,\vec{k}}^\dagger(t_2) \hat{a}_{S,\vec{k}}(t_2) T(t_2, t_1) | 0_{t_1} \rangle_S = {}_H \langle 0_{t_1} | \hat{a}_{H,\vec{k}}^\dagger(t_2) \hat{a}_{H,\vec{k}}(t_2) | 0_{t_1} \rangle_H. \quad (3)$$

Now, using the Bogolubov coefficients we have

$$P_{n,\vec{k}}(t_2, t_1) = |\alpha_{\vec{k}}(t_2, t_1)|^{-2} \left| \frac{\beta_{\vec{k}}(t_2, t_1)}{\alpha_{\vec{k}}(t_2, t_1)} \right|^{2n} = P_{0,\vec{k}}(t_2, t_1) \left( \frac{P_{1,\vec{k}}(t_2, t_1)}{P_{0,\vec{k}}(t_2, t_1)} \right)^n$$

$$N_{\vec{k}}(t_2, t_1) = |\beta_{\vec{k}}(t_2, t_1)|^2 = \frac{P_{1,\vec{k}}(t_2, t_1)}{P_{0,\vec{k}}^2(t_2, t_1)},$$

and, using the “in” and “out” formalism, we obtain

$$P_{0,\vec{k}}(+\infty, -\infty) = |\langle 0_{out,\vec{k}} | 0_{in,\vec{k}} \rangle|^2; \quad N_{\vec{k}}(+\infty, -\infty) = \langle 0_{in} | \hat{a}_{out,\vec{k}}^\dagger \hat{a}_{out,\vec{k}} | 0_{in} \rangle.$$

Therefore, the formula that gives the probability that the vacuum state remains unchanged between  $t_1$  and  $t_2$  is:

$$\begin{aligned} P_0(t_2, t_1) &= \prod_{\vec{k} \in \mathbb{Z}^3} P_{0,\vec{k}}(t_2, t_1) = \prod_{\vec{k} \in \mathbb{Z}^3} |\alpha_{\vec{k} \in \mathbb{Z}^3}(t_2, t_1)|^{-2} = \prod_{\vec{k} \in \mathbb{Z}^3} \frac{1}{1 + N_{\vec{k}}(t_2, t_1)} \\ &= \exp \left( - \sum_{\vec{k} \in \mathbb{Z}^3} \log(1 + N_{\vec{k}}(t_2, t_1)) \right). \end{aligned} \quad (4)$$

In general, if we define ([14],[23])

$$g(x) = \prod_{\vec{k} \in \mathbb{Z}^3} \left( 1 - x \frac{P_{1,\vec{k}}(t_2, t_1)}{P_{0,\vec{k}}(t_2, t_1)} \right)^{-1},$$

then, the probability that  $n$  pairs are created at time  $t_2$ , namely  $P_n(t_2, t_1)$ , is

$$P_n(t_2, t_1) = \frac{1}{n!} \frac{D^n g(0)}{g(1)}.$$

Finally, in the accordance with Feynman [7], the relative probability that a pair is produced, namely  $P_{R,1}(t_2, t_1)$ , is

$$P_{R,1}(t_2, t_1) := \frac{P_1(t_2, t_1)}{P_0(t_2, t_1)} = \sum_{\vec{k} \in \mathbb{Z}^3} \frac{P_{1,\vec{k}}(t_2, t_1)}{P_{0,\vec{k}}(t_2, t_1)} = \sum_{\vec{k} \in \mathbb{Z}^3} \frac{N_{\vec{k}}(t_2, t_1)}{1 + N_{\vec{k}}(t_2, t_1)}.$$

## 2.4 Semiclassical Results

We shall now show the results obtained using the W.K.B. method ([3],[6],[8],[12],[16],[21])

**Theorem 2.1.** *In the semiclassical approach, if we assume that  $\vec{f} \in C_0^\infty(-\infty, \infty)$ , the probability that  $n$  pairs are produced at time  $t$  is [12]*

$$P_n(t, -\infty) = \frac{1}{n!} \left( \frac{\alpha}{64mc^2} \mathcal{E}(t) \right)^n \exp \left( - \frac{\alpha}{64mc^2} \mathcal{E}(t) \right),$$

where,  $\alpha$  is the fine structure constant and  $\mathcal{E}(t) := \frac{L^3}{8\pi c^2} \left| \dot{\vec{f}}(t) \right|^2$  is the energy of the electric field at time  $t$ . It is also given that the average number of produced pairs at time  $t$  is

$$N(t, -\infty) := \sum_{\vec{k} \in \mathbb{Z}^3} N_{\vec{k}}(t, -\infty) = \sum_{n=0}^{\infty} n P_n(t, -\infty) = \frac{\alpha}{64mc^2} \mathcal{E}(t).$$

Consequently, in the semiclassical approximation, pair production is an stochastic Poisson process with expected value  $\frac{\alpha}{64mc^2} \mathcal{E}(t)$ .

**Remark 2.2.** For particles with spin  $\frac{1}{2}$  we have [13]

$$P_n(t, -\infty) = \frac{1}{n!} \left( \frac{3\alpha}{32mc^2} \mathcal{E}(t) \right)^n \exp \left( -\frac{3\alpha}{32mc^2} \mathcal{E}(t) \right).$$

Using the results in [3], we can prove the following [16]:

**Theorem 2.2.** If we assume that the electric field is  $\mathcal{C}^N(\mathbb{R} \setminus \{T\})$  and  $\mathcal{C}^{N-1}$  in  $T$ ; and we suppose, that the field is switched on and off, the average number of pairs produced, when the field is switched off, in the semiclassical approximation, is

$$N(+\infty, -\infty) \sim \frac{\hbar^{2N} \alpha L^3 \|D^{N+1} \vec{f}\|_\infty^2}{(mc^2)^{2N+1} c^2}. \quad (5)$$

For  $N = 0$ , this average number is  $\frac{\alpha}{64mc^2} \frac{1}{8\pi} |\vec{E}(T^+) - \vec{E}(T^-)|^2$ , where  $\vec{E}(T) := \frac{1}{c} \dot{\vec{f}}(T)$  is the electric field at time  $T$ .

In particular, when  $N = 0$ , if we assume  $\vec{E}(T^+) = \vec{0}$  or  $\vec{E}(T^-) = \vec{0}$ , we have

$$N(+\infty, -\infty) = \frac{\alpha}{64mc^2} \mathcal{E}(T)$$

in the semiclassical approximation.

**Remark 2.3.** Using the results obtained in ([6], [21]), and if we assume that the electric field is analytic in  $\mathbb{R}$ , then  $N(+\infty, -\infty)$  is exponentially small in  $\hbar$  ([20], [24]).

### 3 Spinless particles in non-homogeneous fields

In this case, the Klein-Gordon equation is

$$(i\hbar\partial_t - eV(\vec{x}, t))^2 \psi(\vec{x}, t) = (i\hbar c \vec{\nabla} - e\vec{A}(\vec{x}, t))^2 \psi(\vec{x}, t) + m^2 c^4 \psi(\vec{x}, t).$$

The Lagrangian density at time  $t$  is:

$$\mathcal{L}(\vec{x}, t) = |i\hbar\partial_t \psi(\vec{x}, t) - eV(\vec{x}, t) \psi(\vec{x}, t)|^2 - |i\hbar c \vec{\nabla} \psi(\vec{x}, t) - e\vec{A}(\vec{x}, t) \psi(\vec{x}, t)|^2 - m^2 c^4 |\psi(\vec{x}, t)|^2.$$

Let  $\phi(\vec{x}, t) = -i\hbar(i\hbar\partial_t - eV(\vec{x}, t))\psi(\vec{x}, t)$  the momentum. Then, the energy density is

$$\mathcal{E}(\vec{x}, t) = \phi^*(\vec{x}, t) \dot{\psi}(\vec{x}, t) + \dot{\psi}^*(\vec{x}, t) \phi(\vec{x}, t) - \mathcal{L}(\vec{x}, t) = \mathcal{E}_c(\vec{x}, t) + \mathcal{E}_p(\vec{x}, t),$$

where

$$\mathcal{E}_c(\vec{x}, t) = \frac{1}{\hbar^2} |\phi(\vec{x}, t)|^2 + |i\hbar c \vec{\nabla} \psi(\vec{x}, t) - e\vec{A}(\vec{x}, t) \psi(\vec{x}, t)|^2 + m^2 c^4 |\psi(\vec{x}, t)|^2$$

is the kinetic energy density, and  $\mathcal{E}_p(\vec{x}, t) = V(\vec{x}, t) \rho(\vec{x}, t)$  is the potential energy density. We have introduced the electric charge density  $\rho(\vec{x}, t) = -\frac{i}{\hbar} e(\phi(\vec{x}, t) \psi^*(\vec{x}, t) - \psi(\vec{x}, t) \phi^*(\vec{x}, t))$ .

Please note that if we make the change  $\bar{\psi} = \frac{i}{\hbar} \phi$ , then the Klein-Gordon equation is

$$i\hbar\partial_t \begin{pmatrix} \psi(\vec{x}, t) \\ \bar{\psi}(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} eV(\vec{x}, t) & 1 \\ (i\hbar c \vec{\nabla} - e\vec{A}(\vec{x}, t))^2 + m^2 c^4 & eV(\vec{x}, t) \end{pmatrix} \begin{pmatrix} \psi(\vec{x}, t) \\ \bar{\psi}(\vec{x}, t) \end{pmatrix}.$$

To simplify, we suppose that the operator  $(i\hbar c \vec{\nabla} - e\vec{A}(\vec{x}, t))^2 + m^2 c^4$  has a discrete spectrum, and let  $\xi_j(\vec{x}, t)$  be the eigenfunction with eigenvalue  $\lambda(t)$ . We write



$$\psi(\vec{x}, t) = \sum_j A_j(t) \xi_j(\vec{x}, t) \quad \text{and} \quad \phi(\vec{x}, t) = \sum_j B_j(t) \xi_j(\vec{x}, t).$$

Then, we have

$$E_c(t) := \int_{\mathbb{R}^3} \mathcal{E}_c(\vec{x}, t) d\vec{x} = \sum_j \frac{|B_j(t)|^2}{\hbar^2} + \lambda^2(t) |A_j(t)|^2.$$

Now, if we make the canonical change:

$$B_j(t) = \frac{\hbar}{\sqrt{2}} (P_j(t) + i\bar{P}_j(t)) ; \quad A_j(t) = \frac{1}{\hbar\sqrt{2}} (Q_j(t) + i\bar{Q}_j(t)) ,$$

we obtain

$$E_c(t) = \frac{1}{2} \sum_j (P_j^2(t) + \omega_j^2(t) Q_j^2(t)) + (\bar{P}_j^2(t) + \omega_j^2(t) \bar{Q}_j^2(t)) ,$$

where  $\omega_j(t) := \lambda_j(t)/\hbar$ . This is the kinetic energy decomposition in oscillators. We can now quantify these oscillators.

Therefore, the lowest kinetic energy state, i.e., the vacuum state is  $|0_t\rangle = \prod_j |0_{j,t}\rangle$ , where  $|0_{j,t}\rangle = \sqrt{\frac{\omega_j(t)}{\pi\hbar}} \exp\left(-\frac{\omega_j(t)}{2\hbar}(Q_j^2 + \bar{Q}_j^2)\right)$ .

In this case, the creation and annihilation operators are

$$\begin{aligned} \hat{a}_j(t) &= \frac{1}{2\sqrt{\lambda_j(t)}} \left[ (i\hat{P}_j + \omega_j(t)\hat{Q}_j) + i(i\hat{\bar{P}}_j + \omega_j(t)\hat{\bar{Q}}_j) \right] \\ \hat{b}_{-j}^\dagger(t) &= \frac{1}{2\sqrt{\lambda_j(t)}} \left[ (-i\hat{P}_j + \omega_j(t)\hat{Q}_j) + i(-i\hat{\bar{P}}_j + \omega_j(t)\hat{\bar{Q}}_j) \right]. \end{aligned}$$

Using this operators, we have

$$\hat{E}_c(t) = \sum_j \lambda_j(t) \left( a_j^\dagger(t) a_j(t) + b_{-j}^\dagger(t) b_{-j}(t) \right) ; \quad \hat{\rho}(t) = \sum_j \left( a_j^\dagger(t) a_j(t) - b_{-j}^\dagger(t) b_{-j}(t) \right).$$

$$\hat{\psi}(\vec{x}, t) = \sum_j \frac{1}{\sqrt{2\lambda_j(t)}} \left( \hat{a}_j(t) + \hat{b}_{-j}^\dagger(t) \right) \xi_j(\vec{x}, t) \quad \hat{\bar{\psi}}(\vec{x}, t) = \sum_j \sqrt{\frac{\lambda_j(t)}{2}} \left( \hat{a}_j(t) - \hat{b}_{-j}^\dagger(t) \right) \xi_j(\vec{x}, t).$$

Now, the different states that contain a definite number of particles and antiparticles, are eigenfunctions of the kinetic energy operator, and have the following form:

$$\prod_j \frac{(\hat{a}_j^\dagger(t))^{n_j} (\hat{b}_{-j}^\dagger(t))^{m_j}}{\sqrt{n_j!} \sqrt{m_j!}} |0_t\rangle, \quad \text{with } n_j, m_j \in \mathbb{N}.$$

We also have

$$\hat{E}_p(t) := \int_{\mathbb{R}^3} \mathcal{E}_p(\vec{x}, t) d\vec{x} = \int_{\mathbb{R}^3} V(\vec{x}, t) : \hat{\rho}(\vec{x}, t) : d\vec{x},$$

where  $\hat{\rho}(\vec{x}, t) = e(\hat{\psi}(\vec{x}, t) \hat{\bar{\psi}}^*(\vec{x}, t) + \hat{\bar{\psi}}^*(\vec{x}, t) \hat{\psi}(\vec{x}, t))$ , and  $::$  is the normal ordering operator.

Finally, the Schrödinger equation is

$$i\hbar\partial_t|\Psi\rangle = (\hat{E}_c(t) + \hat{E}_p(t))|\Psi\rangle.$$

### 3.1 Example

A very interesting example is the case where  $A_\mu(\vec{x}, t) = (V(\vec{x}, t), 0, 0, 0)$ . Let us suppose that the potential is switched on and off, and assume that it is confined in a box of volume  $L^3$ . Then, in this case, the operators  $\hat{a}_{\vec{k}}, \dots, \hat{b}_{\vec{k}}^\dagger$ , are time independent. Also, we have  $\hat{E}_c = \sum_{\vec{k} \in \mathbb{Z}^3} \lambda_{\vec{k}} (\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \hat{b}_{-\vec{k}}^\dagger \hat{b}_{-\vec{k}})$ , where

$$\lambda_{\vec{k}} = \sqrt{c^2 \left| \frac{2\pi\hbar\vec{k}}{L} \right|^2 + m^2 c^4}.$$

Furthermore

$$\hat{\psi}(\vec{x}) = \sum_{\vec{k} \in \mathbb{Z}^3} \frac{1}{\sqrt{2\lambda_{\vec{k}}}} (\hat{a}_{\vec{k}} + \hat{b}_{-\vec{k}}^\dagger) \xi_{\vec{k}}(\vec{x}); \quad \hat{\bar{\psi}}(\vec{x}) = \sum_{\vec{k} \in \mathbb{Z}^3} \sqrt{\frac{\lambda_{\vec{k}}}{2}} (\hat{a}_{\vec{k}} - \hat{b}_{-\vec{k}}^\dagger) \xi_{\vec{k}}(\vec{x}),$$

where  $\xi_{\vec{k}}(\vec{x}) = \exp\left(i\frac{2\pi}{L}\vec{k} \cdot \vec{x}\right) / L^{\frac{3}{2}}$ .

The Schrödinger equation is  $i\hbar\partial_t|\Psi\rangle = (\hat{E}_c + \hat{E}_p(t))|\Psi\rangle$ . In interaction picture, the equation behaves as follows:  $i\hbar\partial_t|\Psi\rangle_I = \hat{E}_{p,I}(t)|\Psi\rangle_I$ , where  $\hat{E}_{p,I}(t)$  is the potential energy in the interaction picture, which in this case, is

$$\hat{E}_{p,I}(t) = \int V(\vec{x}, t) : \hat{\rho}_I(\vec{x}, t) : d\vec{x}.$$

In this picture, the electric charge density operator is  $\hat{\rho}(\vec{x}, t) = e(\hat{\psi}_I(\vec{x}, t)\hat{\psi}_I^*(\vec{x}, t) + \hat{\psi}_I^*(\vec{x}, t)\hat{\bar{\psi}}_I(\vec{x}, t))$ , with

$$\hat{\psi}_I(\vec{x}, t) = \sum_{\vec{k} \in \mathbb{Z}^3} \frac{1}{\sqrt{2\lambda_{\vec{k}}}} (\hat{a}_{\vec{k}} \xi_{\vec{k}}(\vec{x}, t) + \hat{b}_{\vec{k}}^\dagger \xi_{\vec{k}}^*(\vec{x}, t)); \quad \hat{\bar{\psi}}_I(\vec{x}, t) = \sum_{\vec{k} \in \mathbb{Z}^3} \sqrt{\frac{\lambda_{\vec{k}}}{2}} (\hat{a}_{\vec{k}} \xi_{\vec{k}}(\vec{x}, t) - \hat{b}_{\vec{k}}^\dagger \xi_{\vec{k}}^*(\vec{x}, t)),$$

where  $\xi_{\vec{k}}(\vec{x}, t) = \exp\left(i\left(\frac{2\pi}{L}\vec{k} \cdot \vec{x} - \frac{\lambda_{\vec{k}}}{\hbar}t\right)\right) / L^{\frac{3}{2}}$ .

In the semiclassical limit, we have (see [15]):

$$P_n(t, -\infty) = \frac{1}{n!} \left( \frac{\alpha}{64mc^2} \mathcal{E}(t) \right)^n \exp\left(-\frac{\alpha}{64mc^2} \mathcal{E}(t)\right),$$

where, the energy of the electric field at time  $t$  is now  $\mathcal{E}(t) := \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla V(\vec{x}, t)|^2 d\vec{x}$ .

## 4 Schwinger's Formula for spinless particles

In this section, we deduce and interpret the Schwinger formula for spinless particles. In order to deduce this formula, we consider the potential  $\vec{f}(t) = (0, 0, \chi(t))$ , where

$$\chi(t) = \begin{cases} -cET & \text{if } t < -T \\ cEt & \text{if } -T \leq t \leq T \\ cET & \text{if } t > T, \end{cases}$$

were we have assumed that  $eE > 0$  ([2], [9], [14]). A formal deduction of this formula, is obtained if we take ([12], [14])

$$N_{\vec{k}} = \begin{cases} \exp\left(-\frac{\pi(c^2 p_{\perp}^2 + m^2 c^4)}{\hbar c e E}\right) & \text{if } \left|\frac{2\pi\hbar k_3}{L}\right| \leq eET \\ 0 & \text{if } \left|\frac{2\pi\hbar k_3}{L}\right| > eET, \end{cases} \quad (6)$$

where  $p_{\perp} := \frac{2\pi\hbar}{L}(k_1, k_2)$  and  $N_{\vec{k}} := N_{\vec{k}}(+\infty, -\infty)$ . Then, using formula (4), we have

$$\begin{aligned} |\langle 0_{out} | 0_{in} \rangle|^2 &= \exp\left(-\sum_{\vec{k} \in \mathbb{Z}^3} \log(1 + N_{\vec{k}})\right) = \exp\left(-\sum_{\vec{k} \in \mathbb{Z}^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} N_{\vec{k}}^n\right) \\ &= \exp\left(-\frac{2TL^3 E^2 \alpha}{8\pi^3 \hbar} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp\left(-n \frac{\pi m^2 c^4}{\hbar c e E}\right)\right). \end{aligned}$$

This is in agreement with Schwinger's results ([18], [19], [24], [25]).

In this case, the generating function  $g(x)$  has the following form:

$$g(x) = \exp\left(\frac{2TL^3 E^2 \alpha}{8\pi^3 \hbar} \sum_{n=1}^{\infty} \frac{1}{n^2} ((-1)^{n+1} + (x-1)^n) \exp\left(-n \frac{\pi m^2 c^4}{\hbar c e E}\right)\right).$$

Therefore, using this generating function, we obtain that the average number of pairs produced per unit of volume and per unit of time, is [23]

$$\frac{E^2 \alpha}{8\pi^3 \hbar} \exp\left(-\frac{\pi m^2 c^4}{\hbar c e E}\right),$$

and that the relative probability that a pair is created per unit of volume and time, is [14]

$$\frac{E^2 \alpha}{8\pi^3 \hbar} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \exp\left(-n \frac{\pi m^2 c^4}{\hbar c e E}\right).$$

This is in contrast with Schwinger's interpretation ([10], [19], [24], [25]), who interpreted that

$$\frac{E^2 \alpha}{8\pi^3 \hbar} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp\left(-n \frac{\pi m^2 c^4}{\hbar c e E}\right)$$

is the probability that a pair is created per unit volume and per unit time.

Now we show the way to obtain a rigorous demonstration. Firstly, using semiclassical methods (see [12], [16]), it is easy to prove that

$$\sum_{\substack{\vec{k} \in \mathbb{Z}^3 \\ \frac{2\pi\hbar k_3}{L} \in [eEt_1, eEt_2]}} N_{\vec{k}} \leq 40\pi^3 \frac{L^3}{(2\pi\hbar)^3} \frac{(\hbar c e E)^2}{(mc^2)^2 c^3} c e E (t_2 - t_1). \quad (7)$$

$$\sum_{\substack{\vec{k} \in \mathbb{Z}^3 \\ \frac{2\pi\hbar|\vec{k}_3|}{L} \geq eE \left( T + \sqrt{\frac{mcT}{eE}} \right)}} N_{\vec{k}} \leq \frac{\pi^2 (\hbar ce E)^2 L^3}{8(2\pi\hbar)^3 c^3 mc^2} + \frac{32\pi^2 \hbar^2 (eEc)^3 L^3}{c^3 mc^2 (2\pi\hbar)^3} \frac{T}{(mc^3 eET)^{\frac{1}{4}} (mc^2)^{\frac{1}{2}}}. \quad (8)$$

**Remark 4.1.** Similar bounds are obtained in [9] using another method.

After having obtained these bounds we must study the problem ([2], [9], [16])

$$\ddot{u}_{\vec{k}} + \frac{1}{\hbar^2} \left( c^2 p_{\perp}^2 + c^2 \left( \frac{2\pi\hbar k_3}{L} + eEt \right)^2 + m^2 c^4 \right) u_{\vec{k}} = 0; \quad t \in (-T, T). \quad (9)$$

If we make the following change  $y = \sqrt{\frac{2c}{\hbar e E}} (p_3 + eEt)$ , the differential equation

$$u_{\vec{k}}'' + \left( \frac{1}{4} y^2 - A \right) u_{\vec{k}} = 0, \quad (10)$$

with  $A = \frac{-1}{2eEc\hbar} (c^2 p_{\perp}^2 + m^2 c^4)$ , is obtained.

It is a well-known fact that, the Kummer function allows the construction of an independent set of solutions that verifies when  $y < 0$  ([1], [16], [22]):

$$\varphi_{\vec{k}}^+(y) = \bar{A} B \exp\left(\frac{\pi A}{4}\right) \exp\left(-\frac{i\pi}{8}\right) \left(\frac{y^2}{2}\right)^{-\frac{1}{4}-\frac{i}{2}A} \exp\left(\frac{i}{4}y^2\right) [1 + R_1(A, y^2)] \quad (11)$$

$$\varphi_{\vec{k}}^-(y) = -\bar{A} B \exp\left(\frac{\pi A}{4}\right) \exp\left(\frac{i\pi}{8}\right) \left(\frac{y^2}{2}\right)^{-\frac{1}{4}+\frac{i}{2}A} \exp\left(-\frac{i}{4}y^2\right) [1 + R_2(A, y^2)], \quad (12)$$

with

$$\bar{A} = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{i}{2}A\right) \Gamma\left(\frac{3}{4} + \frac{i}{2}A\right)}; \quad B = \frac{\Gamma\left(\frac{1}{4} + \frac{i}{2}A\right)}{\Gamma\left(\frac{1}{4} - \frac{i}{2}A\right)} + i \frac{\Gamma\left(\frac{3}{4} + \frac{i}{2}A\right)}{\Gamma\left(\frac{3}{4} - \frac{i}{2}A\right)}.$$

Also, when  $y > 0$  verifies:

$$\begin{aligned} \varphi_{\vec{k}}^+(y) = & \exp\left(\frac{\pi A}{4}\right) \left\{ 2\bar{A} \exp\left(\frac{i\pi}{8}\right) \left(\frac{y^2}{2}\right)^{-\frac{1}{4}+\frac{i}{2}A} \exp\left(\frac{-iy^2}{4}\right) [1 + R_3(A, y^2)] \right. \\ & \left. + \bar{A} C \exp\left(\frac{-i\pi}{8}\right) \left(\frac{y^2}{2}\right)^{-\frac{1}{4}-\frac{i}{2}A} \exp\left(\frac{iy^2}{4}\right) [1 + R_4(A, y^2)] \right\} \end{aligned} \quad (13)$$

$$\begin{aligned} \varphi_{\vec{k}}^-(y) = & \exp\left(\frac{\pi A}{4}\right) \left\{ 2i\bar{A}^* \exp\left(\frac{-i\pi}{8}\right) \left(\frac{y^2}{2}\right)^{-\frac{1}{4}-\frac{i}{2}A} \exp\left(\frac{iy^2}{4}\right) [1 + R_5(A, y^2)] \right. \\ & \left. + \bar{A} C \exp\left(\frac{i\pi}{8}\right) \left(\frac{y^2}{2}\right)^{-\frac{1}{4}+\frac{i}{2}A} \exp\left(\frac{-iy^2}{4}\right) [1 + R_6(A, y^2)] \right\}, \end{aligned} \quad (14)$$

with

$$C = \frac{\Gamma\left(\frac{1}{4} + \frac{i}{2}A\right)}{\Gamma\left(\frac{1}{4} - \frac{i}{2}A\right)} - i \frac{\Gamma\left(\frac{3}{4} + \frac{i}{2}A\right)}{\Gamma\left(\frac{3}{4} - \frac{i}{2}A\right)}.$$

In order to prove the Schwinger formula, the key is the following bound:

$$|R_j(A, y^2)| \leq -K \frac{A}{y^2} \exp\left(-\frac{A\pi}{2}\right) \quad j = 1, \dots, 6, \quad (15)$$

where  $K$  is a positive, dimensionless constant that is independent of  $A$  and  $y$ . In order to obtain this bound, we used [22].

**Remark 4.2.** For the derivate we obtain similar expressions to (11),..., (14).

Thus, when  $\frac{2\pi\hbar|k_3|}{eEL} \leq T - \sqrt{\frac{Tmc}{eE}}$ , we have  $y(-T) < 0$  and  $y(T) > 0$ . Therefore, for  $\frac{2\pi\hbar|k_3|}{eEL} \leq T - \sqrt{\frac{Tmc}{eE}}$ , using the bound (15) and formulae (11)...(14), we obtain

$$N_{\vec{k}} = \exp\left(-\frac{\pi}{eEc\hbar}(c^2p_{\perp}^2 + m^2c^4)\right) + F(\vec{p}, T), \quad (16)$$

with

$$|F(\vec{p}, T)| \leq \tilde{K} \frac{c^2p_{\perp}^2 + m^2c^4}{mc^3TeE} \exp\left(-\frac{3\pi}{4eEc\hbar}(c^2p_{\perp}^2 + m^2c^4)\right),$$

where  $\tilde{K}$  is a dimensionless constant that is independent of  $T$ ,  $p_{\perp}$  and  $\hbar$ .

With formulae (7), (8) and (16) we can calculate the average number of pairs produced per unit of volume and unit of time, when  $T \rightarrow \infty$ . We therefore obtain

$$\lim_{T \rightarrow \infty} \sum_{\vec{k} \in \mathbb{Z}^3} \frac{N_{\vec{k}}}{2TL^3} = \frac{E^2\alpha}{8\pi^3\hbar} \exp\left(-\frac{\pi m^2c^4}{\hbar ceE}\right). \quad (17)$$

**Remark 4.3.** A) In ([11], [18], [23]) the authors calculate the quantity

$$\frac{1}{2TL^3} \sum_{\vec{k} \in \mathbb{Z}^3} \lim_{T \rightarrow \infty} |b_{\vec{k}}|^2 = \frac{1}{2T(2\pi\hbar)^3} \int_{\mathbb{R}^3} \exp\left(-\frac{\pi(c^2p_{\perp}^2 + m^2c^4)}{\hbar ceE}\right) d\vec{p},$$

and make the replacement  $\int_{\mathbb{R}} dp_3 \rightarrow 2eET$  in order to obtain the formula (17). Clearly, this argument is meaningless.

B) In the paper [9] the authors wanted to give rigorous proof of the Schwinger formula. However, the step from formula no. 34 to formula no. 37 is not justified. The authors have not obtained the bound (15). For this reason, their demonstration does not have enough mathematical rigor.

On the same basis, we can prove that, when  $T \rightarrow \infty$ , the relative probability that a pair is produced per unit of volume and unit of time, is [16]

$$\lim_{T \rightarrow \infty} \sum_{\vec{k} \in \mathbb{Z}^3} \frac{1}{2TL^3} \frac{|b_{\vec{k}}|^2}{|a_{\vec{k}}|^2} = \frac{E^2\alpha}{8\pi^3\hbar} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \exp\left(-\frac{n\pi m^2c^4}{\hbar ceE}\right). \quad (18)$$

## 5 Conclusions

It is important to realize that, in order to understand the pair production phenomenon in the presence of electromagnetic backgrounds, we must interpret the eigenfunctions of the kinetic energy operator as the states that represent a determinate number of pairs. Therefore, from the Schrödinger equation of the quantized Klein-Gordon field (and, in the case of spin particles, the quantized Dirac field) we can calculate the probability that pairs are created. From this interpretation, it is easy to verify that, when the field is spatially and temporally confined, the probability that the vacuum state remains unchanged, is the exponential of twice the imaginary part of the effective action defined by Schwinger. We have also proved that minus twice the imaginary part of the effective action is not the probability that a pair is created. This is in contrast to Schwinger's interpretation. Finally, we have shown how to prove and interpret the Schwinger formula correctly.

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